the Fibonacci recurrence. Define the ***generating function*** (or ***formal power series***) $\mathcal{F}$ as

$$\begin{aligned}\mathcal{F}(z) &= \sum_{i=0}^{\infty} F_i z^i \\ &= 0 + z + z^2 + 2z^3 + 3z^4 + 5z^5 + 8z^6 + 13z^7 + 21z^8 + \cdots ,\end{aligned}$$

where F_i is the ith Fibonacci number.

a. Show that $\mathcal{F}(z) = z + z\mathcal{F}(z) + z^2\mathcal{F}(z)$.

b. Show that

$$\begin{aligned}\mathcal{F}(z) &= \frac{z}{1 - z - z^2} \\ &= \frac{z}{(1 - \phi z)(1 - \widehat{\phi} z)} \\ &= \frac{1}{\sqrt{5}}\left(\frac{1}{1 - \phi z} - \frac{1}{1 - \widehat{\phi} z}\right) ,\end{aligned}$$

where

$$\phi = \frac{1 + \sqrt{5}}{2} = 1.61803\ldots$$

and

$$\widehat{\phi} = \frac{1 - \sqrt{5}}{2} = -0.61803\ldots .$$

c. Show that

$$\mathcal{F}(z) = \sum_{i=0}^{\infty} \frac{1}{\sqrt{5}}(\phi^i - \widehat{\phi}^i) z^i .$$

d. Prove that $F_i = \phi^i/\sqrt{5}$ for $i > 0$, rounded to the nearest integer. (*Hint:* Observe that $|\widehat{\phi}| < 1$.)

e. Prove that $F_{i+2} \geq \phi^i$ for $i \geq 0$.

4-6 VLSI chip testing

Professor Diogenes has n supposedly identical VLSI[1] chips that in principle are capable of testing each other. The professor's test jig accommodates two chips at

[1] VLSI stands for "very large scale integration," which is the integrated-circuit chip technology used to fabricate most microprocessors today.